Tentamen digitale signaalverwerking voor de sterrenkunde I.

10 februari 2005, $9^{15} - 11^{15}$.

Tijdsduur : 2 uur

1. Vind de Fourier getransformeerde $F(\omega)$ van

a) $$f(t) = \begin{cases} A & 0 < t < T \\ 0 & \text{elders} \end{cases}$$

b) $$f(t) = \begin{cases} 1-t & 0 \le t < 1 \\ 1+t & -1 < t \le 0 \\ 0 & \text{elders} \end{cases}$$

(10 punten)

2. a) Laat zien dat

$$\int_{-\infty}^{+\infty} g(t)\,\dot{\delta}(t)\,dt = -\dot{g}(0)$$

waarbij $\delta(t)$ Dirac's deltafunctie is.

b) Laat zien dat de energie geassocieerd met de deltafunctie oneindig is.

(Hint: één van de manieren waarop men de deltafunctie kan definiëren is:

$$\delta(t) \equiv \lim_{\varepsilon \downarrow 0} \begin{cases} \frac{1}{\varepsilon} & |t| < \frac{\varepsilon}{2} \\ 0 & \text{elders} \end{cases}$$

De energie van een functie f wordt gegeven door $E = \int_{-\infty}^{+\infty} f^2(t)\,dt$)

(10 punten)

3. Als de DFT van een reële rij $x(i), i=0,\ldots,N-1$ gegeven wordt door $X(k), k=0,\ldots,N-1$,
Wat is dan de DFT van x_B:

$$x_B(i) \equiv (x(N-1), x(N-2), \ldots, x(0))$$?

(10 punten)

4. Bepaal de z-transformatie van de volgende rijen, en bepaal hun R.O.C.
$u(n)$ wordt gedefinieerd als: $u(n) = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases}$

a) $f(n) = -\left(\frac{1}{2}\right)^n u(-n-1)$

$g(n) = \left(\frac{1}{2}\right)^n \left(u(n) - u(n-10)\right)$

$h(n) = \delta(n+1) + \delta(n-1)$

(15 punten).

5. a) Bereken de z-transformatie van
$g(n) = \alpha^n u(n)$ en bepaal de R.O.C.
α is hier een constante.

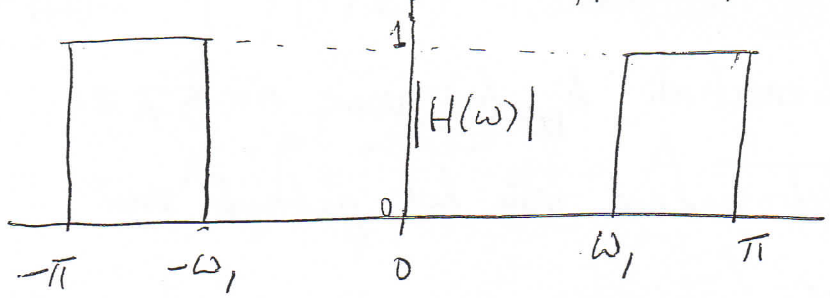
b) Gegeven dat $G(z)$ de z-transformatie is van $g(n)$.
Laat zien dat in de volgende 3 gevallen $H(z)$ de z-transformatie is van $h(n)$.

i) $h(n) = \alpha^n g(n)$ $\qquad H(z) = G(z/\alpha)$

ii) $h(n) = n\, g(n)$ $\qquad H(z) = -z \dfrac{dG(z)}{dz}$

iii) $h(n) = g(n-n_0)$ $\qquad H(z) = z^{-n_0} G(z)$

6. a) Gegeven een ideaal High Pass filter met lineaire fase en groep delay $\alpha = \frac{N-1}{2}$, met N oneven, en cutoff frequentie ω_1



Laat zien dat

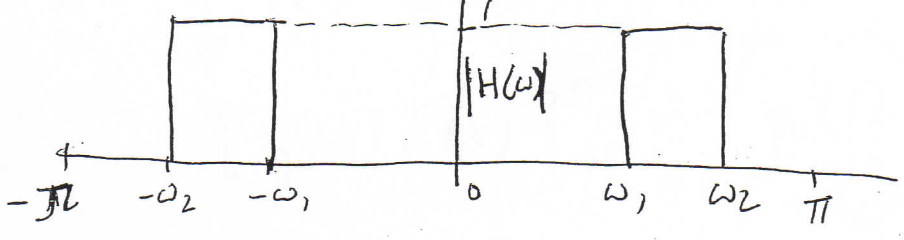
$$\begin{cases} h_{HP}(n) = -\dfrac{\sin \omega_1 (n-\alpha)}{\pi (n-\alpha)} \\ h_{HP}(\alpha) = -\dfrac{\omega_1}{\pi} \end{cases}$$

b) Laat nu $N=7$

Ontwerp nu dit high pass filter met $\omega_1 = 0.6\pi$

(dwz bereken $h(n)$)

c) Een ideaal band pass filter met lineaire fase, groep delay $\alpha = (N-1)/2$, N oneven, ~~en~~ heeft cutoff freq. ω_1 en ω_2



Laat zien dat

$$\begin{cases} h_{BP}(n) = \dfrac{\sin \omega_2 (n-\alpha) - \sin \omega_1 (n-\alpha)}{\pi (n-\alpha)} \\ h_{BP}(\alpha) = \dfrac{\omega_2 - \omega_1}{\pi} \end{cases}$$

d) Met $N=7$ ontwerp dit filter met $\omega_1 = 0.4\pi$ en $\omega_2 = 0.6\pi$.

e) Teken schematisch $h_{HP}(n)$ voor $n = 3, 7, 20$.
Wat is het nadeel van het gebruik van dit filter?

(25 punten)

Answers exam 10 Feb 2005

① $F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-2\pi i \omega t} dt =$

$$= A \int_0^T e^{-2\pi i \omega t} dt = A \cdot \frac{1}{-2\pi i \omega} \left[e^{-2\pi i \omega t} \right]_0^T$$

$$= -\frac{A}{2\pi i \omega} \left(e^{-2\pi i \omega T} - 1 \right)$$

$$= \frac{A}{2\pi i \omega} e^{-\frac{2\pi i \omega T}{2}} \left(e^{+\pi i \omega T} - e^{-\pi i \omega T} \right)$$

$$= \frac{A}{\pi \omega} e^{-\pi i \omega T} \left(\frac{e^{\pi i \omega T} - e^{-\pi i \omega T}}{2i} \right)$$

$$= \frac{A}{\pi \omega} e^{-\pi i \omega T} \sin(\pi \omega T)$$

$$= A T e^{-\pi i \omega T} \frac{\sin(\pi \omega T)}{\pi \omega T}.$$

ⓑ $F(\omega) = \mathrm{sinc}^2(\pi \omega)$ (see Chapter 3)

② a) $\int_{-\infty}^{\infty} g(t) \dot{\delta}(t) dt =$

$$\left[g(t) \delta(t) \right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} g'(t) \delta(t) dt$$

$= 0 - g'(0)$ (property of delta function)

2b) Let $\delta_\varepsilon(t) = \begin{cases} 1/\varepsilon & \text{for } |t| < \varepsilon/2 \\ 0 & \text{elsewhere} \end{cases}$

then $E_\varepsilon = \int_{-\infty}^{\infty} \delta_\varepsilon^2(t)\,dt = \int_{-\varepsilon/2}^{\varepsilon/2} \left(\frac{1}{\varepsilon}\right)^2 dt = \frac{1}{\varepsilon}$

So the energy associated with the delta function

$$E = \lim_{\varepsilon \searrow 0} E_\varepsilon = \text{infinity}.$$

3

The DFT is given by:

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-2\pi i kn/N} = \sum_{n=0}^{N-1} x[n] W_N^{k}$$

Now

$$\tilde{X}[k] = \sum_{n=0}^{N-1} x[N-n-1] e^{-2\pi i kn/N}$$

$$\underset{m=N-n-1}{=} \sum_{m=N-1}^{0} x[m] e^{-2\pi i k(N-1-m)/N}$$

$$= \sum_{m=0}^{N-1} x[m] e^{-2\pi i k(N-m)/N} e^{2\pi i k/N}$$

$$= W_N^{-k} \sum_{m=0}^{N-1} e^{+2\pi i k \cancel{(N-m)} m/N} x[m]$$

For $k=0$ $\quad \tilde{X}[0] = \sum_{m=0}^{N-1} x[m] = X[0]$

For $k \neq 0$ $\quad \tilde{X}[k] = W_N^{-k} \sum_{m=0}^{N-1} x[m] e^{2\pi i km/N}$

$$= W_N^{-k} X[\langle -k \rangle_N]$$

So $\quad \tilde{X}[k] = W_N^{-k} X[\langle -k \rangle_N]$

$g[n] = \alpha^n \mu[n]$

$$G(z) = \sum_{n=-\infty}^{\infty} \alpha^n \mu[n] z^{-n} = \sum_{n=0}^{\infty} \left(\frac{\alpha}{z}\right)^n = \frac{1}{1-\frac{\alpha}{z}}$$

R.O.C: $|\alpha z^{-1}| < 1$ or $|z| > |\alpha|$

b)

i) $$\sum_{n=-\infty}^{\infty} \alpha^n g[n] z^{-n} = \sum_{n=-\infty}^{\infty} g[n] (z/\alpha)^{-n} = G(z/\alpha)$$

ii) $$\frac{dG(z)}{dz} = \frac{d}{dz}\left(\sum_{n=-\infty}^{\infty} g[n] z^{-n}\right) = \sum_{n=-\infty} n z^{-n-1} g[n]$$

So, $$-z\frac{dG(z)}{dz} = \sum_{n=-\infty}^{\infty} n z^{-n} g[n] = \mathcal{Z}(h[n])$$

iii) $$\sum_{n=-\infty}^{\infty} h[n] z^{-n} = \sum_{n=-\infty}^{\infty} g[n-n_0] z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} g[n-n_0] z^{-(n-n_0)} z^{-n_0} = z^{-n_0} \sum_{n=-\infty}^{\infty} g[n-n_0] z^{-(n-n}$$

$$= z^{-n_0} G(z)$$

(4) a) $F(z) = \sum_{n=-\infty}^{\infty} -\left(\frac{1}{2}\right)^n \mu(-n-1) z^{-n}$

$= \sum_{n=-\infty}^{-1} \left(-\left(\frac{1}{2}\right)^n z^{-n}\right)$

$= -\sum_{n=-\infty}^{-1} (2z)^{-n} \underset{\substack{\uparrow \\ m=-n}}{=} -\sum_{m=1}^{\infty} (2z)^m =$

$-\left(\sum_{m=0}^{\infty} (2z)^m - 1\right) =$

$-\left(\frac{1}{1-2z} - 1\right) = -\left(\frac{1-(1-2z)}{1-2z}\right) = \frac{2z}{1-2z}$

ROC: $\{z \mid |z| < \frac{1}{2}\}$

b) $G(z) = \sum_{n=0}^{9} \left(\frac{1}{2}\right)^n z^{-n} = \sum_{n=0}^{\infty} (2z)^{-n} - \sum_{n=10}^{\infty} (2z)^{-n}$

$= \sum_{n=0}^{\infty} (2z)^{-n} - (2z)^{-10} \sum_{n=0}^{\infty} (2z)^{-n}$

$= \frac{1}{1-(2z)^{-1}} - \frac{1}{1-(2z)^{-1}} \cdot \frac{1}{(2z)^{10}}$

$= \frac{1-(2z)^{-10}}{1-(2z)^{-1}}$ ROC: $\{z \mid |z| > \frac{1}{2}\}$

c) $H(z) = \sum_{n=-\infty}^{\infty} \left(\delta[n+1] + \delta[n-1]\right) z^{-n} = z + z^{-1}$

R.O.C $= \{z \mid z \neq 0\}$

(6) The phase is linear; ~~so~~ group delay $\alpha = \frac{N-1}{2}$, so

a) $$H_{HP}(e^{i\omega}) = e^{i(\alpha\omega+\beta)}\tilde{H}_{HP}(\omega)$$

where $\tilde{H}(\omega)$ is a real function of ω.

We take $\beta=0$, since this simplifies the calculations and the choice of β does not influence the result.

$$h_{HP}[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_{HP}(e^{i\omega})e^{i\omega n}\,d\omega$$

$$= \frac{1}{2\pi}\int_{\omega_1}^{\pi} e^{-i\alpha\omega}\cdot e^{i\omega n}\,d\omega + \frac{1}{2\pi}\int_{-\pi}^{-\omega_1} e^{-i\alpha\omega}\cdot e^{i\omega n}\,d\omega$$

(if $n\neq\alpha$)

$$= \frac{1}{2\pi}\left(\frac{1}{i(n-\alpha)}\left[e^{i\omega(n-\alpha)}\right]_{\omega_1}^{\pi} + \frac{1}{i(n-\alpha)}\left[e^{i\omega(n-\alpha)}\right]_{-\pi}^{-\omega_1}\right)$$

$$= \frac{1}{2\pi i(n-\alpha)}\left(\cancel{e^{i\pi(n-\alpha)}} - e^{i\omega_1(n-\alpha)} + e^{-i\omega_1(n-\alpha)} - \cancel{e^{-i\pi(n-\alpha)}}\right)$$

$$= -\frac{1}{2\pi i(n-\alpha)}\left(e^{i\omega_1(n-\alpha)} - e^{-i\omega_1(n-\alpha)}\right)$$

$$= -\frac{\sin(\omega_1(n-\alpha))}{\pi(n-\alpha)}$$

If $n=\alpha$:

$$h_{HP}[n] = \frac{1}{2\pi}\int_{\omega_1}^{\pi} 1\cdot d\omega + \frac{1}{2\pi}\int_{-\pi}^{-\omega_1} 1\,d\omega = -\frac{\omega_1}{\pi}.$$

b) Fill in $\omega_1 = 0.6\pi$, $N=7$.

$$h_{HP}[n] = -\frac{\sin(0.6\pi(n-3))}{\pi(n-3)} \qquad (n=0,\dots,6)$$

c.) $$h_{BP}[n] = \frac{1}{2\pi}\int_{-\omega_2}^{-\omega_1} e^{-i\alpha\omega} e^{i\omega n}\, d\omega + \frac{1}{2\pi}\int_{\omega_1}^{\omega_2} e^{-i\alpha\omega} e^{i\omega n}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\omega_2}^{-\omega_1} e^{i\omega(n-\alpha)}\, d\omega + \frac{1}{2\pi}\int_{\omega_1}^{\omega_2} e^{i\omega(n-\alpha)}\, d\omega$$

$$\underset{n\neq\alpha}{=} \frac{1}{2\pi}\,\frac{1}{i(n-\alpha)}\left[e^{i\omega(n-\alpha)}\right]_{-\omega_2}^{-\omega_1} + \frac{1}{2\pi i(n-\alpha)}\left[e^{i\omega(n-\alpha)}\right]_{\omega_1}^{\omega_2}$$

$$= \frac{1}{2i\pi(n-\alpha)}\left(e^{-i\omega_1(n-\alpha)} - e^{-i\omega_2(n-\alpha)} + e^{i\omega_2(n-\alpha)} - e^{i\omega_1(n-\alpha)}\right)$$

$$= \frac{\sin(\omega_2(n-\alpha)) - \sin(\omega_1(n-\alpha))}{\pi(n-\alpha)}$$

Similar to b)

if $n=\alpha$ $\quad h_{BP}[\alpha] = \dfrac{\omega_2-\omega_1}{\pi}$

d) $N=7$

$$h_{BP}[n] = \frac{\sin(0.6\pi(n-3)) - \sin 0.4\pi(n-3))}{\pi(n-3)}$$

$(n=0,\dots,6)$

e) Plot these terms yourself.

The problem is that there is a lot of power at high n-values, so that cutting off at some value of n will introduce oscillations in $\tilde{H}(\omega)$